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Strength-Based Reliability and Fracture Assessment of Fuzzy Mechanical and Structural Systems

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In most mechanical and structural systems, a certain amount of uncertainty exists in both strength and load values. The uncertainty of these values must be taken into consideration when quantifying the safety and reliability of a system. Fuzzy set theory has been used for the description and analysis of such systems. A means for comparing fuzzy strength and load values is needed to determine the safety and reliability of fuzzy mechanical and structural systems. The concepts of fuzzy bounds and subsethood are used here to define a safety index for mechanical or structural systems based on fuzzy load and strength values. Similarly, a criticality index for fracture damage assessment is defined based on fuzzy stress intensity factors. The concept of a fuzzy factor of safety is also introduced. The proposed safety and criticality indices and the fuzzy factor of safety could provide useful measures for rating designs in terms of strength-based reliability and damage tolerance. The application of the proposed measures in the assessment of damage as reported by online structural damage detection systems is also suggested.

I. Introduction

In most mechanical and structural systems, a certain amount of uncertainty exists in both strength and load values. The uncertain aspects of these values must be taken into consideration when quantifying the safety and reliability of a system. If the strength S and load L are random in nature with known probability density functions, the probability of failure is the probability of the load exceeding the strength of the component, as shown in Fig. 1 (overlapping or interference area of the strength and load probability density functions, f_S and f_L). Conversely, reliability can be defined as the probability of strength exceeding the load for all possible values of the load:

$$R = \int \mathrm{d}R = \int_{-\infty}^{\infty} f_L(l)\left[\int_{l}^{\infty} f_S(s)\,\mathrm{d}s\right]\mathrm{d}l$$
$$= \int_{-\infty}^{\infty} f_L(l)[1 - F_S(l)]\,\mathrm{d}l \qquad (1)$$

where $F_S(l)$ is the probability distribution function of the strength S (Ref. 1).

When subjective judgment or experience is to be combined with hard data to obtain estimates similar to those obtained from the conventional statistical methods, the Bayesian approach can be used.[2] The verification of new design by using Bayes' theorem was discussed with consideration of a missile system by Blodgett et al.[3] A multilevel design involving uncertainty was realized by using a compromise decision support problem formulation based on Bayesian statistic by Vadde et al.[4]

However, uncertainty in structural or mechanical systems is not always probabilistic in nature. Most systems are described by fuzzy information that is vague, imprecise, qualitative, linguistic, or incomplete. The fuzzy or imprecise information may be present in the geometry, material properties, applied loads, or boundary conditions of a system. Fuzzy set theory has been applied for the representation and analysis of such systems. Stresses computed in the analysis of fuzzy structural or mechanical systems are presented as fuzzy numbers. In the area of damage detection in smart structures, the assessment of identified damage may involve the analysis of fuzzy residual strength or the analysis of fracture described in fuzzy terms. Here, also, the strength and load values to be compared are fuzzy values. Therefore, means for comparing fuzzy strength and load values are needed to quantify the safety and reliability of fuzzy mechanical and structural systems. The concepts of fuzzy bounds and subsethood, which measure the degree to which a fuzzy value is within a bound, are used to define a safety index for mechanical and structural systems based on fuzzy load and strength values. A criticality index for damage assessment of fuzzy fracture in structures is also defined. The concept of a fuzzy factor of safety based on fuzzy load and strength values is also introduced.

Fuzzy set theory was initiated by Zadeh.[5] Recent applications of the subject to various scientific areas, such as artificial intelligence, image processing, speech recognition, biological and medical sciences, decision theory, economics, geography, sociology, psychology, linguistics, and semiotics, have shown that fuzzy set theory is indeed a useful tool for the quantification of impreciseness and vagueness present in many real-life problems. Most engineering applications of fuzzy set theory have been related to controls, decision making, and optimization. Several papers have been written on the applications of fuzzy set theory to the area of structural analysis. Brown and Yao[6,7] discussed the use of fuzzy set theory in structural design. The fuzzy optimum design of structures was studied by Yuan and Quan.[8] The application of fuzzy methodologies to the multiobjective optimization of mechanical and structural systems was presented by Rao.[9,10] The use of nonlinear membership functions in the optimization of engineering systems was proposed and demonstrated by Dhingra et al.[11] The fuzzy random vibration with fuzzy parameters, with application to aseismic structures, was considered by Wang and Ou.[12] A fuzzy finite element approach for the analysis of imprecisely defined systems was introduced by Rao and Sawyer.[13]

The importance of membership functions and a comparative discussion of different learning methods for fuzzy inference systems was presented by Lotfi and Tsoi.[14] Law and Antonsson[15] developed a technique known as the method of imprecision for modeling, representing and manipulating fuzzy uncertainties associated with the design process decision making and compared it with other techniques, such as utility theory.

The fuzzy and Bayesian approaches were combined in the study of reliability of existing structures by Chou and Yuan.[16] A mixed fuzzy-probabilistic approach was developed by Shih and Wangsawidjaja[17] for multiobjective optimization with random variables. A comparative study of the various uncertainty management techniques was made by Saffiotti et al.[18]

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II. Fuzzy Safe Limits Based on Material Properties

The effective assessment of a machine or structure is highly dependent on the ability to predict under which circumstances failure is likely to occur in the load carrying members. The failure of a machine or structure is established based on a failure theory such as the von Mises criterion and the permissible stress of the material. In applying these strength-based failure criteria, an effective stress value that characterizes the state of stress is determined. This characteristic stress is then compared to the yield or fracture strength of the material to assess the safety and integrity of a structural system. In fracture mechanics, a stress intensity factor K is defined to characterize the severity of a crack as a function of crack size, stress, and geometry. A given material can resist a crack without any brittle fracture as long as the value of K is below a critical value K_c, the fracture toughness of the material. The stress intensity induced by a given load is compared to the fracture toughness when determining the criticality of a crack in a structure or machine component. The uncertainty in the material properties must be quantified in a form that is consistent with the induced loads so that comparison can be made.

A. Fuzzy Set Representation of Probabilistic Mechanical Properties

The mechanical behavior of engineering materials is probabilistic in nature due to random differences in chemistry and processing. Strength data for materials that exhibit plastic behavior generally follow a normal distribution.[19] The statistical characteristics of tensile yield and ultimate strengths of some common engineering materials are listed in Table 1. For brittle materials, a type III minimum value distribution is applicable.[1] Fracture toughness values for most materials can also be assumed to be normally distributed.[19] Fracture toughness is a function of material thickness; thus, data are usually tabulated for specimens of varying thickness. Statistical parameters for the plane stress fracture toughness K_{Ic} of some engineering materials are listed in Table 2.

In determining the reliability and safety of fuzzy structural and mechanical systems, the probabilistic uncertainty in material properties must be represented in terms of fuzzy sets. The relationship between fuzzy set theory and probability theory continues to be debated in the mathematics community. Kosko, a prominent fuzzy theorist, has argued that "fuzziness contains probability as a special case" (see Ref. 20). For practical applications, Civanlar and Trussell[21] proposed a set of criteria and methods for constructing membership functions for statistically based fuzzy sets. A statistically based fuzzy set is one whose membership function is based on the probability density function of a defining feature. One such criterion is the possibility–probability consistency principle that can be stated as follows: "The degree of possibility of an element is greater than or equal to its degree of probability."[21] A simple method of defining a membership function based on statistical data normalizes the probability density function as shown in Fig. 2 from which the membership function can be determined as

$$\mu(x) = \lambda p(x) \tag{2}$$

where

$$\lambda = \frac{1}{\max[p(x)]} \tag{3}$$

and $p(x)$ is the probability density function or its estimate derived from the histogram of the feature x used for defining the fuzzy set. This approach can be used to construct a membership function for a fuzzy set that represents the strength or fracture toughness of a material based on statistical data. The probability density functions and the corresponding fuzzy numbers for the yield strength of an aluminum alloy and a medium carbon steel are shown in Figs. 3 and 4, respectively. Probabilistic and fuzzy representations for the fracture toughness of a 2024-T3 aluminum alloy sheet are given in Fig. 5. In Figs. 3–5, the material properties are assumed to follow normal distribution.

Table 1 Strength data for various engineering materials[19]

	Tensile yield, ksi		Ultimate tensile, ksi	
Material	Mean value	Standard deviation	Mean value	Standard deviation
2014 Aluminum forging	63.0	2.23	70.0	1.89
2024-T3 Aluminum alloy	——	——	67.0	0.93
1006 Carbon steel	35.7	0.80	48.3	0.52
1035 Carbon steel	49.5	5.36	86.2	3.92
Type 302 stainless steel	42.4	3.53	90.4	3.01
Type 304 stainless steel	85.0	4.14	37.9	3.76
Inconel 718	141.2	2.34	173.0	2.39
A-286 alloy	102.2	2.25	153.0	2.87

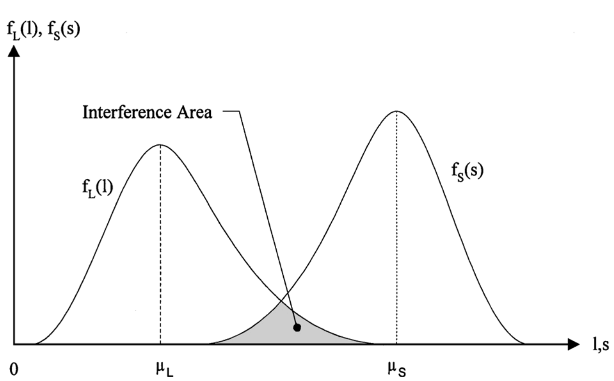


Fig. 1 Strength-based reliability theory based on the interference of probability density functions.[1]

Table 2 Fracture toughness data for various engineering materials[19]

			Fracture toughness, ksi√in.	
Material	Thickness, in.	Width, in.	Mean value	Standard deviation
2014-T6 Aluminum alloy	0.063	18.00	65.18	1.39
2024-T3 Aluminum alloy	0.080	20.00	83.44	2.07
7075-T6 Aluminum alloy	0.040	6.00	37.42	12.41
Inconel 718	0.025	4.00	161.42	8.57

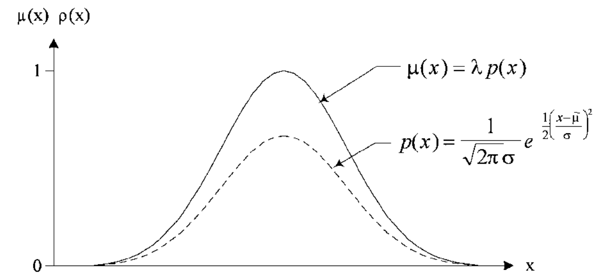


Fig. 2 Construction of a membership function $\mu(x)$ based on a Gaussian probability distribution function $p(x)$.

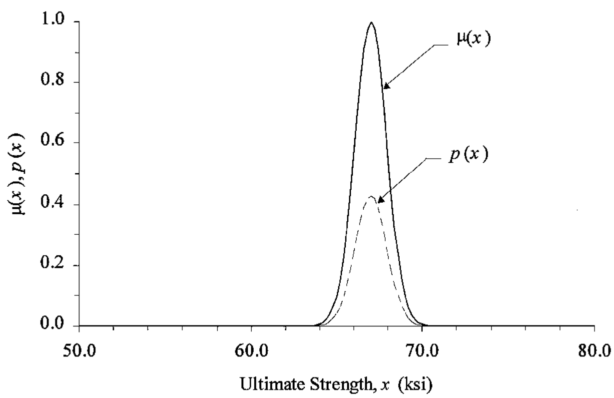


Fig. 3 Probability density function $p(x)$ and a fuzzy number $\mu(x)$ representation for the ultimate strength of 2024-T3 aluminum alloy.

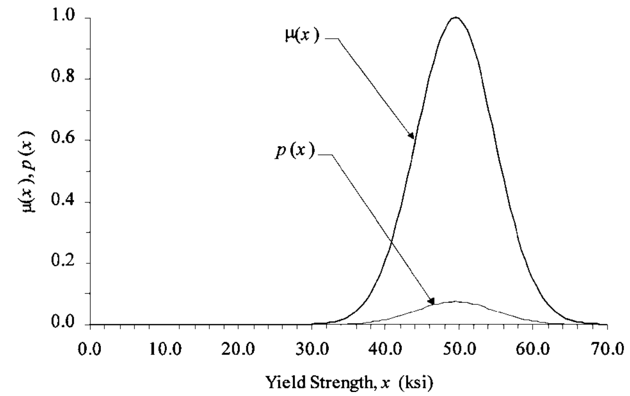


Fig. 4 Probability density function $p(x)$ and a fuzzy number $\mu(x)$ representation for the yield strength of AISI 1035 carbon steel.

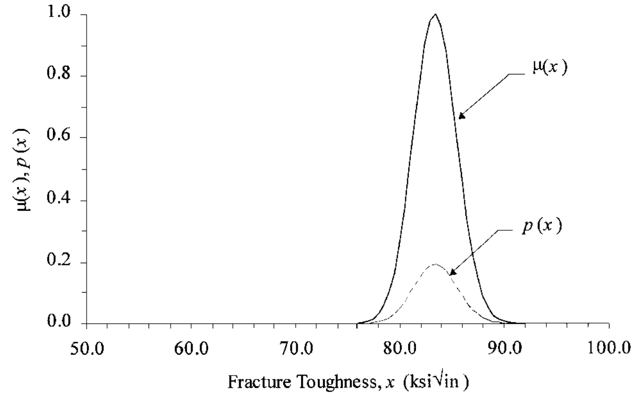


Fig. 5 Probability density function $p(x)$ and a fuzzy number $\mu(x)$ representation for the fracture toughness of 2024-T3 aluminum sheet of thickness 0.08 in.

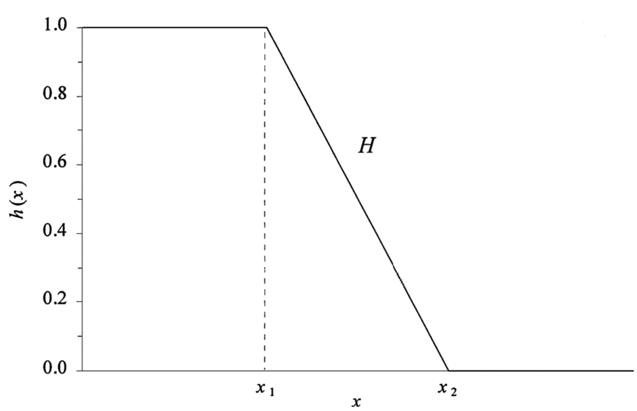


Fig. 6 Fuzzy upper bound.

B. Fuzzy Bounds and Safe Limits

A fuzzy upper bound is defined by Kaufmann and Gupta[22] as a function $h(x)$ such that

$$\begin{aligned} h(x) &= 1, \quad x \le x_1 \\ &= \text{a monotonically decreasing function}, \quad x_1 \le x \le x_2 \\ &= 0, \quad x \ge x_2 \end{aligned} \tag{4}$$

This function can be represented by a fuzzy subset $H \subset R$, where $h(x)$ is a membership function, as shown in Fig. 6. Consider, for example, the fuzzy set of safe loads that may be applied to a structural component shown in Fig. 7. This membership function defines the fuzzy upper bound of loads that are considered safe. Loads less than 60 ksi are completely safe (membership value of one). Load values between 60 and 90 ksi are safe to varying degrees. Loads above 90 ksi are not safe (membership grade of zero). The fuzzy upper bound that defines the safe limit of loads (stresses) that can be induced in a mechanical or structural component can be derived from fuzzy strength values.

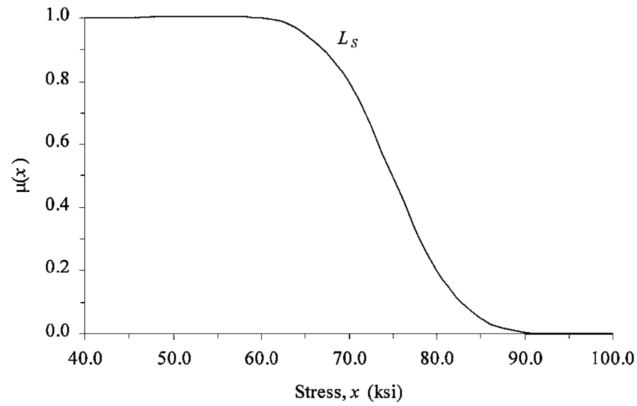


Fig. 7 Membership function for a safe load fuzzy upper bound.

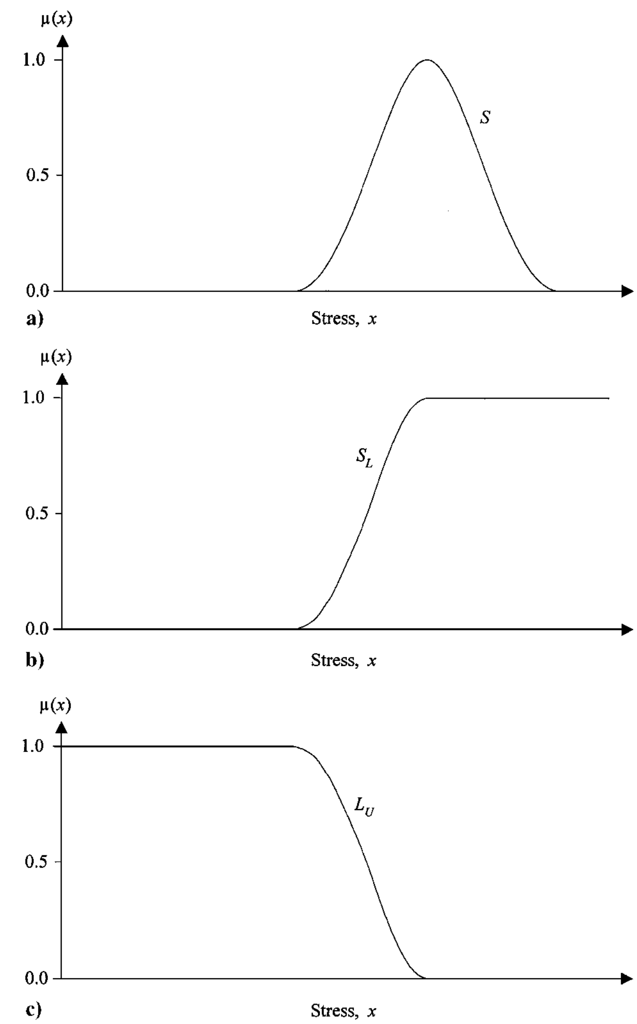


Fig. 8 Formulation of a fuzzy safe upper load limit begins with a) fuzzy strength, which is then used to define b) the membership function S_L for the set of values that are equal to or exceed the strength S, and then c) the complement of the lower bound of the strength defines the safe upper limit of load values L_U.

First, consider strength as a fuzzy set S, shown in Fig. 8a. This set may be derived from the statistical data of a material strength characteristic such as yield strength or fracture toughness. Next, a fuzzy set S_L based on the lower bound values of strength is defined, as shown in Fig. 8b, to indicate the set of values of stress that are equal to or greater than the strength. The complement of this set denotes the set of stress values that are lower than the strength. This fuzzy bound, shown in Fig. 8c, describes the safe upper limit of load as it denotes the set of values that are lower than the strength. The

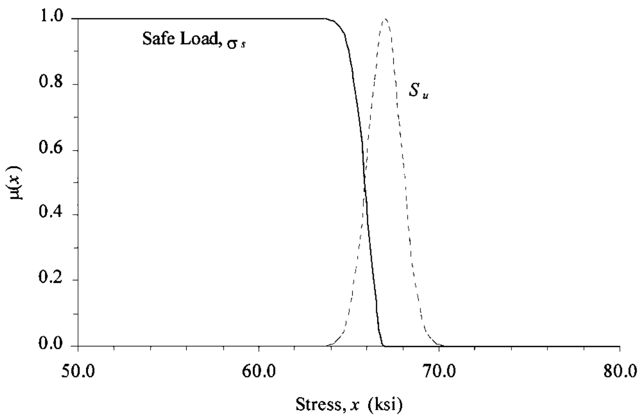


Fig. 9 Safe upper bound σ_s for loads based on the fuzzy ultimate strength S_u of 2024-T3 aluminum alloy.

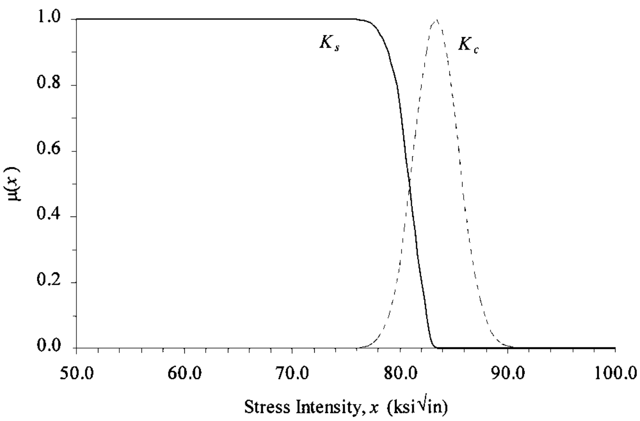


Fig. 10 Safe upper bound for stress intensity K_s based on the fuzzy fracture toughness K_c of 2024-T3 aluminum sheet of thickness 0.08 in.

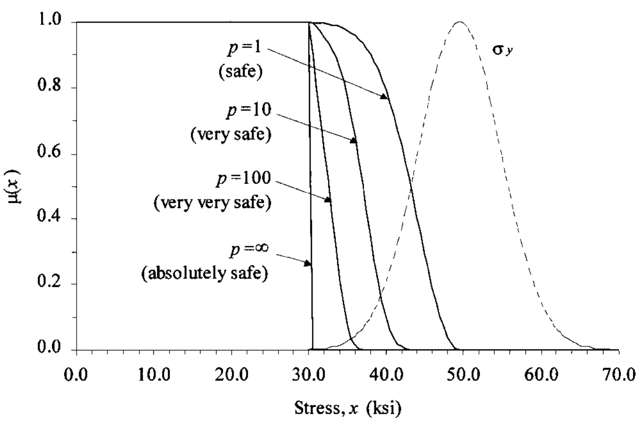


Fig. 11 Safe load upper bound of varying degrees based on the yield strength of AISI 1035 carbon steel.

safe upper bound based on the membership function for the fuzzy ultimate strength of 2024-T3 aluminum alloy is shown in Fig. 9. A fuzzy bound that represents the safe upper limit for stress intensity in the 2024-T3 aluminum sheet based on the membership function of fuzzy fracture toughness is shown in Fig. 10.

C. Linguistic Hedges and Degrees of Safety

A fuzzy upper bound such as the one that denotes a safe load may be modified with linguistic hedges to represent restricted sets such as very safe load, very very safe load, and absolutely safe load. For a given safe upper bound L_S with a membership function μ_{L_S}, restricted sets L_S^* may be defined such that

$$\mu_{L_s^*} = (\mu_{L_s})^p, \qquad 1 \le p \le \infty \tag{5}$$

Membership functions corresponding to various values of p are shown in Fig. 11. When $p = 1$, the upper bound on $\mu_{L_s^*}$ denotes the safe load μ_{L_s} determined from the fuzzy strength parameter. With increasing p, this fuzzy upper bound is contracted thereby restricting the definition of a safe load. As $p \to \infty$, the set defined by $\mu_{L_s^*}$ approaches a crisp set that does not overlap the fuzzy strength set. This crisp set is the set of absolutely safe loads because, based on the membership functions defined, there is no possibility of a load in this set exceeding the strength. Such linguistic modifiers and the associated values of p may be used to denote varying degrees of criticality in specific applications, much like the use of factors of safety in design.

III. Measures of Safety and Damage Criticality Based on Fuzzy Subsethood

A. Fuzzy Subsethood and Supersethood

Subsethood is the degree to which one fuzzy set contains another set. The fuzzy subsethood theorem introduced by Kosko[23] defines a measure of subsethood. The degree to which a discrete fuzzy set A is a subset of the fuzzy set B is given by

$$S(A, B) = 1 - \frac{\sum_{i=1}^{n} \max[0, m_A(x_i) - m_B(x_i)]}{M(A)} \tag{6}$$

where n is the number of elements in the universe of discourse, m_A and m_B are the membership functions of A and B, respectively, and $M(A)$ is the size or cardinality of A given by

$$M(A) = \sum_{i=1}^{n} m_A(x_i) \tag{7}$$

If B is a fuzzy upper bound, $S(A, B)$ is a measure of the degree to which A is within this bound.

Consider, for example, the fuzzy upper bound on the safe load shown in Fig. 7. This fuzzy set can be represented by the fit vector

$$\boldsymbol{L_S} = (1, 1, 0.95, 0.8, 0.5, 0.2, 0.05, 0, 0, 0) \tag{8}$$

where the universe of discourse is

$$X = \{40, 60, 65, 70, 75, 80, 85, 90, 95, 100\} \tag{9}$$

If a fuzzy load $\boldsymbol{L}$ of about 75 ksi is defined by the fit vector

$$\boldsymbol{L} = (0, 0.25, 0.5, 0.75, 1, 0.75, 0.5, 0.25, 0, 0) \tag{10}$$

the degree of subsethood is given by

$$S(\boldsymbol{L}, \boldsymbol{L_s}) = 1 - \frac{(0.5 + 0.55 + 0.45 + 0.25)}{(0.25 + 0.5 + 0.75 + 1 + 0.75 + 0.5 + 0.25)} = 0.48 \tag{11}$$

On a scale of zero to one, this is the degree to which the load $\boldsymbol{L}$ is within the fuzzy bound of a safe load $\boldsymbol{L_s}$.

The opposite or complement of subsethood is called supersethood $\bar{S}$, defined by[23]

$$\bar{S}(A, B) = 1 - S(A, B) = \frac{\sum_{i=1}^{n} \max[0, m_A(x_i) - m_B(x_i)]}{M(A)} \tag{12}$$

If B is a fuzzy upper bound, $\bar{S}(A, B)$ denotes a measure of the degree to which A exceeds this bound. In the preceding example, the degree to which the fuzzy load $\boldsymbol{L}$ is a superset of the fuzzy bound $\boldsymbol{L_s}$ is given by

$$\bar{S}(\boldsymbol{L}, \boldsymbol{L_s}) = 1 - S(\boldsymbol{L}, \boldsymbol{L_s}) = 1 - 0.48 = 0.52 \tag{13}$$

B. Safety Index Based on Fuzzy Structural Analysis

In evaluating the reliability of a fuzzy machine or structural component, a characteristic stress is compared to the strength of the material. The probabilistic material strength defines a safe fuzzy limit for the induced stress. The degree of subsethood of the fuzzy stress with respect to the safe limit can be evaluated. Thus, a safety index can be defined as

$$SI = S(\sigma, \sigma_s) = 1 - \frac{\sum_{i=1}^{n} \max\left[0, m_\sigma(x_i) - m_{\sigma_s}(x_i)\right]}{M(\sigma)} \tag{14}$$

where σ is the computed fuzzy stress and σ_s is the fuzzy bound defining the safe limit of the stress.

As an example, consider the 15-bar truss shown in Fig. 12. For a fuzzy load P, described by the membership function of Fig. 13, the fuzzy stresses developed in the truss can be evaluated using the fuzzy finite element method. Considering the truss in an undamaged condition and a damaged state where element 1 is severed, the respective deterministic stiffness matrices can be found as

$$\boldsymbol{K}_0 = \begin{bmatrix} 172 & 0 & 0 & 0 & -63.4 & 0 & -22.4 & 22.4 & 0 & 0 & 0 & 0 \\ & 108 & 0 & -63.4 & 0 & 0 & 22.4 & -22.4 & 0 & 0 & 0 & 0 \\ & & 172 & 0 & -22.4 & -22.4 & -63.4 & 0 & 0 & 0 & 0 & 0 \\ & & & 108 & -22.4 & -22.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 172 & 0 & 0 & 0 & -63.4 & 0 & -22.4 & 22.4 \\ & & & & & 108 & 0 & -63.4 & 0 & 0 & 22.4 & -22.4 \\ & & & & & & 172 & 0 & -22.4 & -22.4 & -63.4 & 0 \\ & & & & & & & 108 & -22.4 & -22.4 & 0 & 0 \\ & & \text{sym} & & & & & & 85.8 & 22.4 & 0 & 0 \\ & & & & & & & & & 85.8 & 0 & -63.4 \\ & & & & & & & & & & 85.8 & -22.4 \\ & & & & & & & & & & & 85.8 \end{bmatrix} N/M \tag{15}$$

and

$$\boldsymbol{K}_d = \begin{bmatrix} 108 & 0 & 0 & 0 & -63.4 & 0 & -22.4 & 22.4 & 0 & 0 & 0 & 0 \\ & 108 & 0 & -63.4 & 0 & 0 & 22.4 & -22.4 & 0 & 0 & 0 & 0 \\ & & 172 & 0 & -22.4 & -22.4 & -63.4 & 0 & 0 & 0 & 0 & 0 \\ & & & 108 & -22.4 & -22.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 172 & 0 & 0 & 0 & -63.4 & 0 & -22.4 & 22.4 \\ & & & & & 108 & 0 & -63.4 & 0 & 0 & 22.4 & -22.4 \\ & & & & & & 172 & 0 & -22.4 & -22.4 & -63.4 & 0 \\ & & & & & & & 108 & -22.4 & -22.4 & 0 & 0 \\ & & \text{sym} & & & & & & 85.8 & 22.4 & 0 & 0 \\ & & & & & & & & & 85.8 & 0 & -63.4 \\ & & & & & & & & & & 85.8 & -22.4 \\ & & & & & & & & & & & 85.8 \end{bmatrix} N/M \tag{16}$$

The fuzzy finite element equations given by

$$[\boldsymbol{K}] \begin{Bmatrix} u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

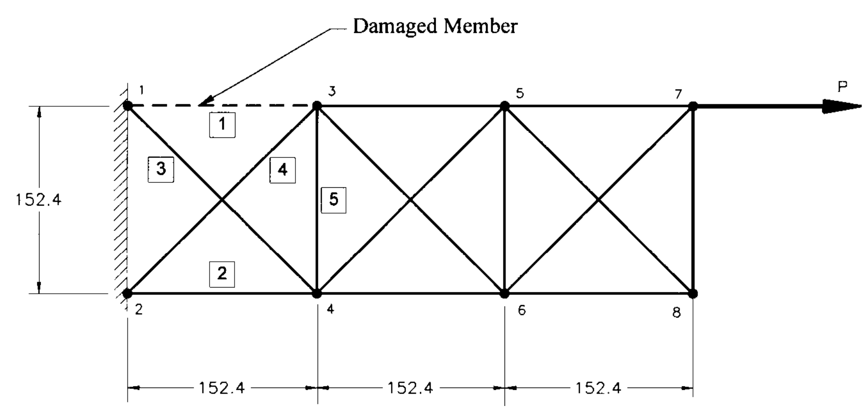


Fig. 12 Example: 15-bar planar truss, all dimensions in centimeters and all components are 2024-T3 aluminum with an elastic modulus of 69.0 $\times 10^3$ MPa and cross-section area 1.4 $\times 10^{-3}$ m^2.

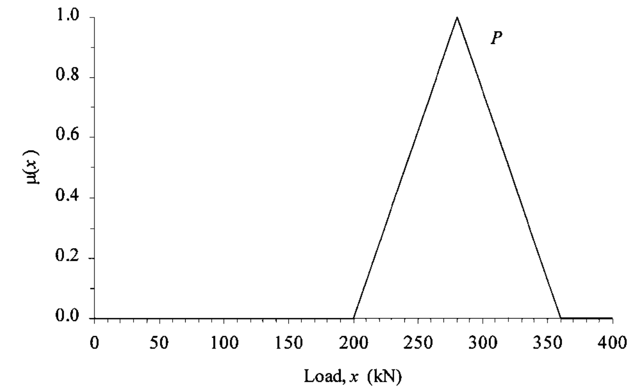


Fig. 13 Fuzzy load P applied in the truss example.

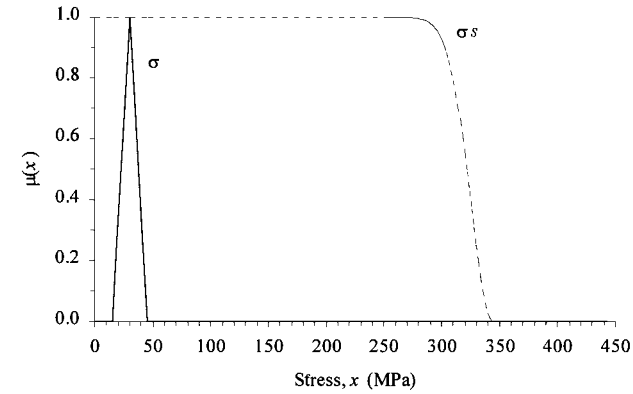


Fig. 14 Fuzzy stress and safe upper bound for loads in elements 3 and 4 of the example truss in an undamaged state; safety index for this example is 1.0.

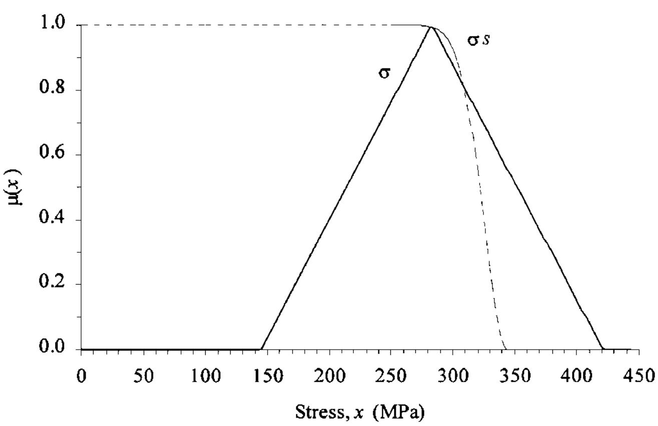


Fig. 15 Fuzzy stress and safe upper bound for loads in elements 3 and 4 of the example truss with damage; safety index for this example is 0.67.

can be solved to find the fuzzy displacement vector $\boldsymbol{u}$. Based on the displacement of node 4, the fuzzy stress induced in element 3 can be computed using fuzzy arithmetic in each case.

For the case of the undamaged truss, the fuzzy stress is shown in Fig. 14 along with a fuzzy safe upper bound corresponding to the ultimate tensile strength of 2024-T3 aluminum. The safety index of members 3 and 4, in this case, is given by

$$SI = S(\sigma, \sigma_s) = 1 - [0.0/\,M(\sigma)] = 1.0 \tag{18}$$

The safety index of 1.0 indicates that the load σ is completely within the bounds of the safe load limit σ_s, which can be observed clearly in Fig. 14. For the same load condition, when element 1 is severed (damaged truss), the fuzzy stresses in elements 3 and 4 increase to the level shown in Fig. 15. In this case, the safety index drops to

$$SI = S(\sigma, \sigma_s) = 1 - (16.3/\,47.6) = 0.67 \tag{19}$$

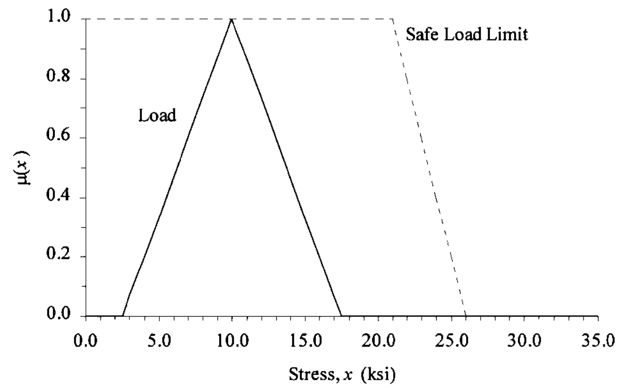


Fig. 16 Fuzzy load of about 10 ksi compared to a fuzzy safe load limit for a mild carbon steel; safety index for this example is 1.0.

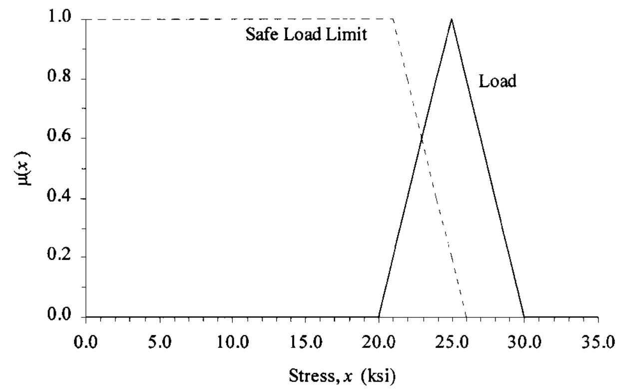


Fig. 17 Fuzzy load of about 25 ksi compared to a fuzzy safe load limit for a mild carbon steel; safety index for this example is 0.36.

As a second example, consider the fuzzy stress induced in a stepped bar.[13] In this case, the load is a uniaxial tension that induces a fuzzy maximum principal stress of about 10 ksi. In applying the maximum shear stress theory for the uniaxial case, this fuzzy principal stress is compared to the safe load limit based on the maximum yield strength of the material. Using the safe limit of mild carbon steel, shown in Fig. 16, the safety index can be seen to be clearly 1.0 because the fuzzy load is completely below the fuzzy safe load limit. If the fuzzy load is increased to about 25 ksi, as shown in Fig. 17, the safety index reduces to 0.36 indicating a strong possibility of the load exceeding the yield strength of the material.

The safety index, defined in the interval [0, 1], provides a relative measure of safety in a design with respect to strength and loading. An index of 1.0 indicates the highest degree of safety, that is, with all uncertainties considered, the possible strength levels exceed all of the possible stresses computed in the analysis. At the other end of the scale, an index of 0.0 indicates that the system is unsafe for all possible load and strength configurations as defined by the fuzzy parameters. Thus, the safety index provides a useful measure that can rate the safety of mechanical or structural components within the transition interval from absolutely acceptable to unacceptable with respect to strength and loading.

C. Criticality Index for Fuzzy Damage

Because of the unavoidable uncertainties inherent in damaged structures and detection systems, the reported crack lengths and, hence, the computed stress intensity factors will be fuzzy quantities. In assessing the criticality of a crack, the fuzzy stress intensity factor should be compared to a safe limit based on the fracture toughness of the material. The supersethood measure of the reduced stress intensity relative to the fracture toughness can be used to define a criticality index for fracture damage as

$$CI = \bar{S}(K, K_s) = \frac{\sum_{i=1}^{n} \max\left[0, m_K(x_i) - m_{K_s}(x_i)\right]}{M(K)} \tag{20}$$

where K is the fuzzy stress intensity, K_s is the fuzzy upper limit for the stress intensity, and n is the number of elements in the discrete

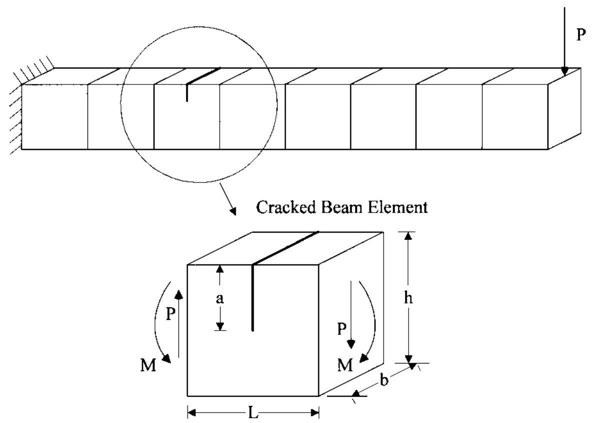


Fig. 18 Cracked beam element parameters.

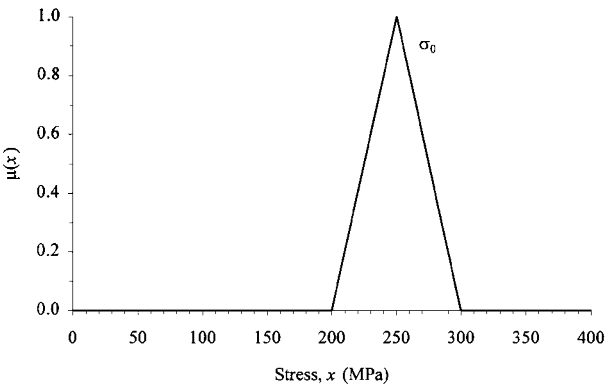
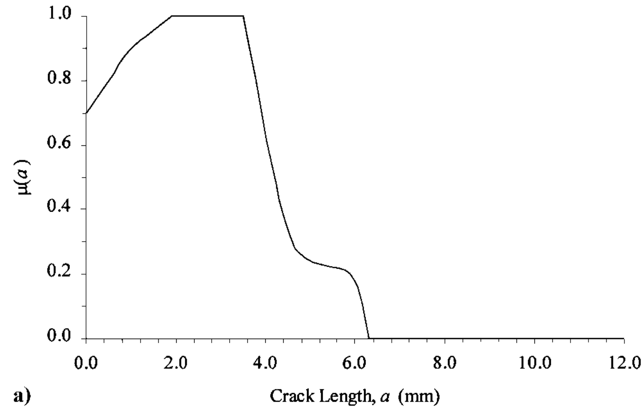


Fig. 19 Fuzzy stress σ_0 assumed in the cracked beam examples.



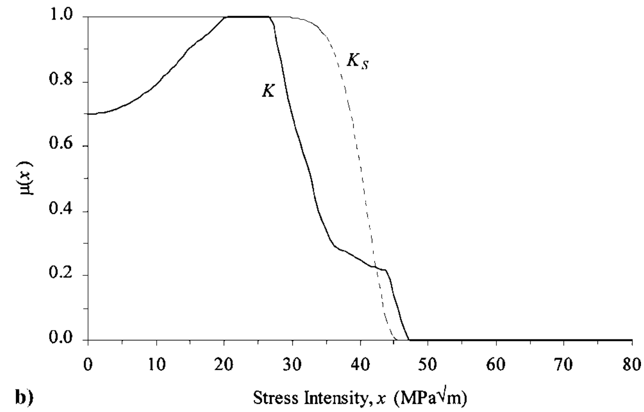


Fig. 20 Safe upper limit K_s comparison: a) fuzzy crack length and b) corresponding fuzzy stress intensity *K*.

universe of discourse. This criticality index for fracture damage, defined in the interval [0, 1], provides a relative measure of severity of a crack in a structure or mechanical component with respect to a particular load condition and fracture toughness. An index of 1.0 indicates the highest degree of severity. At the other end of the scale, an index of 0.0 indicates the damage is not critical in that the stress intensity induced by the crack is below the safe limit as defined by the fuzzy fracture toughness of the material.

As an example, consider the edge cracked beam shown in Fig. 18 for which the relationship between crack length a and mode I stress intensity is given by

$$K_{\mathrm{I}} = \sigma_0 \sqrt{\pi a} F_{\mathrm{I}}(s) \tag{21}$$

where σ_0 is the remotely applied stress and F_{I} is the geometric shape factor

$$F_{\mathrm{I}}(s) = \sqrt{\frac{2}{\pi s} \tan\left(\frac{\pi s}{2}\right)} \frac{0.923 + 0.199[1 - \sin(\pi s/2)]^4}{\cos(\pi s/2)} \tag{22}$$

where $s = a/h$ (Ref. 24). Given a fuzzy crack length a and a fuzzy load σ_0, fuzzy arithmetic can be applied in evaluating Eq. (22) to find the fuzzy stress intensity factor K_{I}. For the cracked beam, the fuzzy stress σ_0 is assumed to be as indicated in Fig. 19. The fuzzy crack lengths and the corresponding stress intensity factors for two different cases are shown in Figs. 20 and 21. A safe upper bound for stress intensity K_s, based on the fracture toughness of AISI 4130 steel, is also plotted in Figs. 20 and 21. The criticality index for the first case is given by

$$CI = \bar{S}(K, K_s) = \frac{0.776}{56.2} = 0.014 \tag{23}$$

This relatively low criticality index indicates that, although it is possible that the stress intensity will exceed the fracture toughness, it is rather unlikely. In the second case, where the crack length and stress intensity have larger values, the criticality index is given by

$$CI = \bar{S}(K, K_s) = 33.9/53.1 = 0.638 \tag{24}$$

The damage in this case is much more severe as there is a greater possibility of stress intensity exceeding the fracture toughness under the given fuzzy load conditions (Fig. 21).

Because there is uncertainty in determining the stress intensity induced by a crack and the fracture toughness of materials, there is a transition from what can be considered absolutely critical and absolutely not critical. The criticality index can be used in evaluating damage in slow crack growth structures as reported by online detection systems. It can also be used in damage tolerance studies of new design concepts. In such applications the criticality index could provide a useful measure that can rate particular designs with various damage conditions in the transition interval between absolutely acceptable and unacceptable in a fashion similar to the proposed safety index.

IV. Fuzzy Factor of Safety

The factor of safety commonly used to evaluate the safety of a machine or structural component is defined as

$$n = \sigma_U / \sigma \tag{25}$$

where σ_U is some limiting strength value and σ is the working load. If the strength and load are fuzzy, a fuzzy factor of safety can be computed based on fuzzy arithmetic techniques and the extension principle. To illustrate the concept of fuzzy factor of safety, the fuzzy strength of AISI 1035 steel is plotted along with a particular fuzzy load in Fig. 22a. The corresponding fuzzy factor of safety, computed from Eq. (25), is shown in Fig. 22b. This possibility distribution indicates that factor of safety has a modal value of $n = 1.65$, and the critical value $n = 1$ has a possibility of less than 0.05. The possibility that the load could exceed the strength is also indicated by the small area of overlap of the load and strength distributions (Fig. 22a).

As a second example, consider the 15-bar truss shown in Fig. 12. When the truss is subjected to the given fuzzy load, the fuzzy stress σ induced in members 3 and 4 will be as shown in Fig. 23 for the undamaged and damaged states. The fuzzy ultimate strength of 2024-T3 aluminum alloy is also shown in Fig. 23. The computed

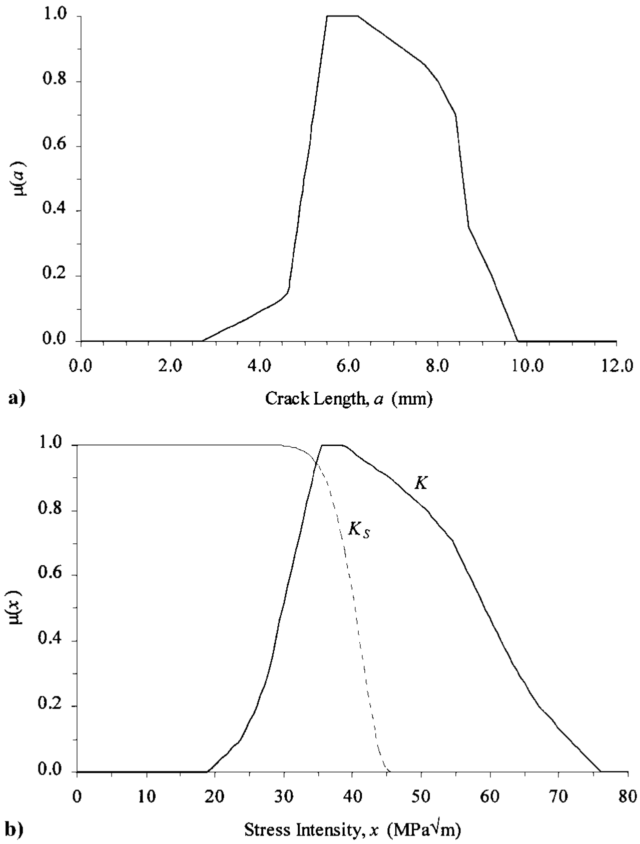


Fig. 21 Safe upper limit K_s comparison: a) fuzzy crack length and b) corresponding fuzzy stress intensity K.

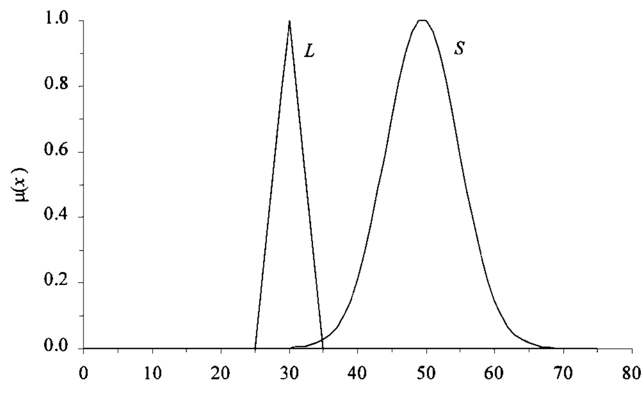


Fig. 22a Fuzzy load of about 30 ksi; the fuzzy yield strength of AISI 1035 steel.

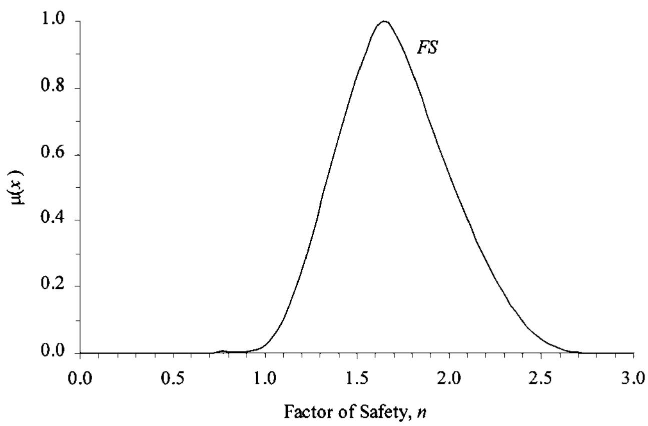


Fig. 22b Corresponding fuzzy factor of safety.

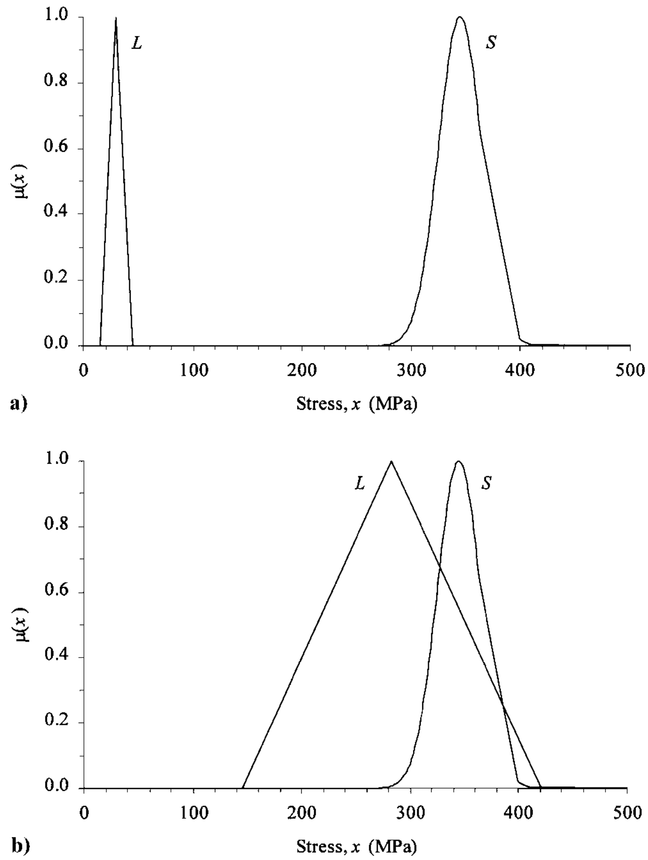


Fig. 23 Fuzzy load and strength values for members 3 and 4 of the 15-bar truss in a) undamaged and b) damaged condition.

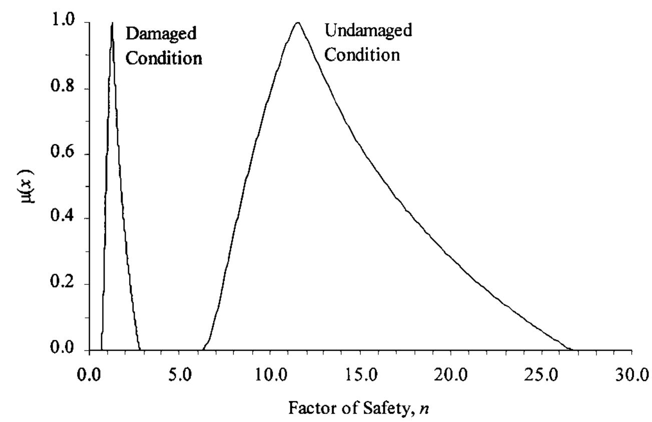


Fig. 24 Fuzzy factors of safety for the 15-bar truss example.

fuzzy factors of safety for these cases are shown in Fig. 24. As expected, the factor of safety in the damaged condition is less than that for the undamaged structure due to the higher load. The possibility distribution for the undamaged case indicates that n has a modal value of 5.75 (corresponding to $\mu = 1$) and the critical value of $n = 1$ has a possibility of 0.0. The modal value for the factor of safety in the damaged structure is $n = 1.3$, and the critical value of $n = 1$ has a possibility of 0.4.

From these examples, it is interesting to note that an increase in the uncertainty of the load does not necessarily result in an increase in the fuzziness of the factor of safety. The spread of the fuzzy factor of safety for the damaged case is clearly narrower than that for the undamaged case. To understand why the fuzziness of the factor of safety decreases as the load increases, we need to consider the limit. As the lower bound on the load tends to infinity, both the upper and lower bounds on the factor of safety approach a value of zero. Thus, the factor of safety approaches a crisp zero with no uncertainty as the load increases.

The factor of safety represented as a fuzzy number provides a greater insight into the possible safety of a member in comparison to a single estimated value representing a deterministic quantity based on assumed crisp values of strength and load. The safety index, presented earlier, measures the degree to which a load exceeds some safe bound. If the load is completely within the bound, it does not provide a measure of the distance from the bound. The membership function for a fuzzy factor of safety provides a possibility measure for the critical value of $n = 1$, and it also provides information relative to the degree of safety over the complete range of safe and unsafe loads.

V. Conclusion

A safety index for quantifying the strength-based reliability of fuzzy structural and mechanical systems is defined. A fracture damage criticality index for rating the severity of a crack based on a detected fuzzy crack length and stress intensity factor is also proposed. Both indices are derived from the concept of a fuzzy upper bound and subsethood, which is a measure of the degree to which a fuzzy number falls within the fuzzy bound. The notion of a fuzzy factor of safety based on fuzzy strength and load values is also introduced. The conversion of probabilistic material properties into fuzzy number representations is investigated. Methods for generating fuzzy bounds that define safe upper limits for loads based on fuzzy material properties are also suggested.

The newly defined safety index provides a means for assessing the safety of structural systems within the transition interval between absolute acceptability and unacceptability with respect to strength and loading. The proposed fuzzy factor of safety can also be used to assess the safety of load carrying members under fuzzy conditions. The proposed criticality index can be used in evaluating damage of structures involving slow crack growth. These measures can be used in damage tolerance studies of new design concepts or in evaluating the severity of damage as reported by online detection system in a smart structure. The safety and criticality indices and the fuzzy factor of safety could provide useful measures for rating particular designs in terms of strength-based reliability and damage tolerance.

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